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DECIDABILITY OF THE INTERVAL TEMPORAL LOGIC $\mathsf{AB\bar{B}}$ OVER THE NATURAL NUMBERS

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Abstract. In this paper, we focus our attention on the interval temporal logic of the Allen's relations "meets", "begins", and "begun by" ($\mathsf{AB\bar{B}}$ for short), interpreted over natural numbers. We first introduce the logic and we show that it is expressive enough to model distinctive interval properties, such as accomplishment conditions, to capture basic modalities of point-based temporal logic, such as the until operator, and to encode relevant metric constraints. Then, we prove that the satisfiability problem for $\mathsf{AB\bar{B}}$ over natural numbers is decidable by providing a small model theorem based on an original contraction method. Finally, we prove the EXPSPACE-completeness of the problem.

1. Introduction

Interval temporal logics are modal logics that allow one to represent and to reason about time intervals. It is well known that, on a linear ordering, one among thirteen different binary relations may hold between any pair of intervals, namely, "ends", "during", "begins", "overlaps", "meets", "before", together with their inverses, and the relation "equals" (the so-called Allen's relations [1])[1]. Allen's relations give rise to respective unary modal operators, thus defining the modal logic of time intervals HS introduced by Halpern and Shoham in [12]. Some of these modal operators are actually definable in terms of others; in particular, if singleton intervals are included in the structure, it suffices to choose as basic the modalities corresponding to the relations "begins" B and "ends" E, and their transposes $\mathsf{\bar{B}}$, $\mathsf{\bar{E}}$. HS turns out to be highly undecidable under very weak assumptions on the class of

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[1]We do not consider here the case of ternary relations. Amongst the multitude of *ternary* relations among intervals there is one of particular importance, which corresponds to the binary operation of concatenation of meeting intervals. The logic of such a ternary interval relation has been investigated by Venema in [18]. A systematic analysis of its fragments has been recently given by Hodkinson et al. [13].

interval structures over which its formulas are interpreted [12]. In particular, undecidability holds for any class of interval structures over linear orderings that contains at least one linear ordering with an infinite ascending or descending chain, thus including the natural time flows $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$, and $\mathbb{R}$. In [14], Lodaya sharpens the undecidability of HS showing that the two modalities B, E suffice for undecidability over dense linear orderings (in fact, the result applies to the class of all linear orderings [11]). Even though HS is very natural and the meaning of its operators is quite intuitive (unlike what happens with other temporal logics), for a long time such sweeping undecidability results have discouraged the search for practical applications and further investigations in the field. A renewed interest in interval temporal logics has been recently stimulated by the identification of some decidable fragments of HS, whose decidability does not depend on simplifying semantic assumptions such as locality and homogeneity [11]. This is the case with the fragments $\mathsf{B\bar{B}}$, $\mathsf{E\bar{E}}$ (logics of the "begins/begun by" and "ends/ended by" relations) [11], A, $\mathsf{A\bar{A}}$ (logics of temporal neighborhood) [10], and D, $\mathsf{D\bar{D}}$ (logics of the subinterval/superinterval relations) [3, 15].

In this paper, we focus our attention on the product logic $\mathsf{AB\bar{B}}$, obtained from the join of $\mathsf{B\bar{B}}$ and A (the case of $\mathsf{\bar{A}E\bar{E}}$ is fully symmetric), interpreted over the linear order N of the natural numbers (or a finite prefix of it). The decidability of $\mathsf{B\bar{B}}$ can be proved by translating it into the point-based propositional temporal logic of linear time LTL with temporal modalities F (sometime in the future) and P (sometime in the past), which has the finite (pseudo-)model property and is decidable, e.g., [9]. In general, such a reduction to point-based temporal logics does not work: formulas of interval temporal logics are evaluated over pairs of points and translate into binary relations. For instance, this is the case with A. Unlike the case of $\mathsf{B\bar{B}}$, when dealing with A one cannot abstract way from the left endpoint of intervals, because contradictory formulas may hold over intervals with the same right endpoint and a different left endpoint. The decidability of $\mathsf{A\bar{A}}$, and thus that of its fragment A, over various classes of linear orderings has been proved by Bresolin et al. by reducing its satisfiability problem to that of the two-variable fragment of first-order logic over the same classes of structures [4], whose decidability has been proved by Otto in [16]. Optimal tableau methods for A with respect to various classes of interval structures can be found in [6, 7]. A decidable metric extension of A over the natural numbers has been proposed in [8]. A number of undecidable extensions of A, and $\mathsf{A\bar{A}}$, have been given in [2, 5].

$\mathsf{AB\bar{B}}$ retains the simplicity of its constituents $\mathsf{B\bar{B}}$ and A, but it improves a lot on their expressive power (as we shall show, such an increase in expressiveness is achieved at the cost of an increase in complexity). First, it allows one to express assertions that may be true at certain intervals, but at no subinterval of them, such as the conditions of accomplishment. Moreover, it makes it possible to easily encode the until operator of point-based temporal logic (this is possible neither with $\mathsf{B\bar{B}}$ nor with A). Finally, meaningful metric constraints about the length of intervals can be expressed in $\mathsf{AB\bar{B}}$, that is, one can constrain an interval to be at least (resp., at most, exactly) k points long. We prove the decidability of $\mathsf{AB\bar{B}}$ interpreted over N by providing a small model theorem based on an original contraction method. To prove it, we take advantage of a natural (equivalent) interpretation of $\mathsf{AB\bar{B}}$ formulas over grid-like structures based on a bijection between the set of intervals over N and (a suitable subset of) the set of points of the $\mathsf{N} \times \mathsf{N}$ grid. In addition, we prove that the satisfiability problem for $\mathsf{AB\bar{B}}$ is EXPSPACE-complete (that for A is NEXPTIME-complete). In the proof of hardness, we use a reduction from the exponential-corridor tiling problem.

The paper is organized as follows. In Section 2 we introduce $\mathsf{AB\bar{B}}$. In Section 3, we prove the decidability of its satisfiability problem. We first describe the application of the contraction method to finite models and then we generalize it to infinite ones. In Section 4 we deal with computational complexity issues. Conclusions provide an assessment of the work and outline future research directions. Missing proofs are reported in the Appendix.

2. The interval temporal logic $\mathsf{AB\bar{B}}$

In this section, we briefly introduce syntax and semantics of the logic $\mathsf{AB\bar{B}}$, which features three modal operators $\langle A\rangle$, $\langle B\rangle$, and $\langle \bar{B}\rangle$ corresponding to the three Allen's relations A ("meets"), B ("begins"), and $\bar{B}$ ("begun by"), respectively. We show that $\mathsf{AB\bar{B}}$ is expressive enough to capture the notion of accomplishment, to define the standard until operator of point-based temporal logics, and to encode metric conditions. Then, we introduce the basic notions of atom, type, and dependency. We conclude the section by providing an alternative interpretation of $\mathsf{AB\bar{B}}$ over labeled grid-like structures.

2.1. Syntax and semantics

Given a set $\mathcal{P}rop$ of propositional variables, formulas of $\mathsf{AB\bar{B}}$ are built up from $\mathcal{P}rop$ using the boolean connectives $\neg$ and $\vee$ and the unary modal operators $\langle A\rangle$, $\langle B\rangle$, $\langle \bar{B}\rangle$. As usual, we shall take advantage of shorthands like $\varphi_1 \wedge \varphi_2 = \neg(\neg\varphi_1 \vee \neg\varphi_2)$, $[A]\varphi = \neg\langle A\rangle\neg\varphi$, $[B]\varphi = \neg\langle B\rangle\neg\varphi$, etc. Hereafter, we denote by $|\varphi|$ the size of φ.

We interpret formulas of $\mathsf{AB\bar{B}}$ in interval temporal structures over natural numbers endowed with the relations "meets", "begins", and "begun by". Precisely, we identify any given ordinal $N \leqslant \omega$ with the prefix of length N of the linear order of the natural numbers and we accordingly define $\mathbb{I}_N$ as the set of all non-singleton closed intervals $[x, y]$, with $x, y \in N$ and $x < y$. For any pair of intervals $[x, y], [x', y'] \in \mathbb{I}_N$, the Allen's relations "meets" A, "begins" B, and "begun by" $\bar{B}$ are defined as follows (note that $\bar{B}$ is the inverse relation of B):

- **"meets" relation:** $[x, y]\ A\ [x', y']$ iff $y = x'$;
- **"begins" relation:** $[x, y]\ B\ [x', y']$ iff $x = x'$ and $y' < y$;
- **"begun by" relation:** $[x, y]\ \bar{B}\ [x', y']$ iff $x = x'$ and $y < y'$.

Given an *interval structure* $\mathcal{S} = (\mathbb{I}_N, A, B, \bar{B}, \sigma)$, where $\sigma : \mathbb{I}_N \to \mathscr{P}(\mathcal{P}rop)$ is a labeling function that maps intervals in $\mathbb{I}_N$ to sets of propositional variables, and an initial interval I, we define the semantics of an $\mathsf{AB\bar{B}}$ formula as follows:

- $\mathcal{S}, I \models a$ iff $a \in \sigma(I)$, for any $a \in \mathcal{P}rop$;
- $\mathcal{S}, I \models \neg\varphi$ iff $\mathcal{S}, I \not\models \varphi$;
- $\mathcal{S}, I \models \varphi_1 \vee \varphi_2$ iff $\mathcal{S}, I \models \varphi_1$ or $\mathcal{S}, I \models \varphi_2$;
- for every relation $R \in \{A, B, \bar{B}\}$, $\mathcal{S}, I \models \langle R\rangle\varphi$ iff there is an interval $J \in \mathbb{I}_N$ such that $I\ R\ J$ and $\mathcal{S}, J \models \varphi$.

Given an interval structure $\mathcal{S}$ and a formula φ, we say that $\mathcal{S}$ *satisfies* φ (and hence φ is *satisfiable*) if there is an interval I in $\mathcal{S}$ such that $\mathcal{S}, I \models \varphi$. Accordingly, we define the *satisfiability problem* for $\mathsf{AB\bar{B}}$ as the problem of establishing whether a given $\mathsf{AB\bar{B}}$-formula φ is satisfiable.

We conclude the section with some examples that account for $\mathsf{AB\bar{B}}$ expressive power. The first one shows how to encode in $\mathsf{AB\bar{B}}$ conditions of accomplishment (think of formula φ as the assertion: "Mr. Jones flew from Venice to Nancy"): $\langle A\rangle\big(\varphi \wedge [B](\neg\varphi \wedge [A]\neg\varphi)\big)$. Formulas of point-based temporal logics of the form $\psi\ \mathsf{U}\ \varphi$, using the standard until operator, can be encoded in $\mathsf{AB\bar{B}}$ (where atomic intervals are two-point intervals) as follows: $\langle A\rangle\big([B]\bot \wedge \varphi\big) \vee \langle A\rangle\big(\langle A\rangle([B]\bot \wedge \varphi) \wedge [B](\langle A\rangle([B]\bot \wedge \psi))\big)$. Finally, metric conditions like: "φ holds over a right neighbor interval of length greater than k (resp., less than k, equal to k)" can be captured by the following $\mathsf{AB\bar{B}}$ formula: $\langle A\rangle\big(\varphi \wedge \langle B\rangle^{k}\top\big)$ (resp., $\langle A\rangle\big(\varphi \wedge [B]^{k}\bot\big)$, $\langle A\rangle\big(\varphi \wedge [B]^{k}\bot \wedge \langle B\rangle^{k-1}\top\big)$)[2].

2.2. Atoms, types, and dependencies

Let $\mathcal{S} = (\mathbb{I}_N, A, B, \bar{B}, \sigma)$ be an interval structure and φ be a formula of $\mathsf{AB\bar{B}}$. In the sequel, we shall compare intervals in $\mathcal{S}$ with respect to the set of subformulas of φ they satisfy. To do that, we introduce the key notions of φ-atom, φ-type, φ-cluster, and φ-shading.

First of all, we define the *closure* $\mathcal{C}l(\varphi)$ of φ as the set of all subformulas of φ and of their negations (we identify $\neg\neg\alpha$ with α, $\neg\langle A\rangle\alpha$ with $[A]\neg\alpha$, etc.). For technical reasons, we also introduce the *extended closure* $\mathcal{C}l^+(\varphi)$, which is defined as the set of all formulas in $\mathcal{C}l(\varphi)$ plus all formulas of the forms $\langle R\rangle\alpha$ and $\neg\langle R\rangle\alpha$, with $R \in \{A, B, \bar{B}\}$ and $\alpha \in \mathcal{C}l(\varphi)$.

A φ-*atom* is any non-empty set $F \subseteq \mathcal{C}l^+(\varphi)$ such that (i) for every $\alpha \in \mathcal{C}l^+(\varphi)$, we have $\alpha \in F$ iff $\neg\alpha \notin F$ and (ii) for every $\gamma = \alpha \vee \beta \in \mathcal{C}l^+(\varphi)$, we have $\gamma \in F$ iff $\alpha \in F$ or $\beta \in F$ (intuitively, a φ-*atom* is a maximal *locally consistent* set of formulas chosen from $\mathcal{C}l^+(\varphi)$). Note that the cardinalities of both sets $\mathcal{C}l(\varphi)$ and $\mathcal{C}l^+(\varphi)$ are *linear* in the number $|\varphi|$ of subformulas of φ, while the number of φ-atoms is *at most exponential* in $|\varphi|$ (precisely, we have $|\mathcal{C}l(\varphi)| = 2|\varphi|$, $|\mathcal{C}l^+(\varphi)| = 14|\varphi|$, and there are at most $2^{7|\varphi|}$ distinct atoms).

We also associate with each interval $I \in \mathcal{S}$ the set of all formulas $\alpha \in \mathcal{C}l^+(\varphi)$ such that $\mathcal{S}, I \vDash \alpha$. Such a set is called φ-*type* of I and it is denoted by $\mathcal{T}ype_{\mathcal{S}}(I)$. We have that every φ-type is a φ-atom, but not vice versa. Hereafter, we shall omit the argument φ, thus calling a φ-atom (resp., a φ-type) simply an atom (resp., a type).

Given an atom F, we denote by $\mathcal{O}bs(F)$ the set of all *observables* of F, namely, the formulas $\alpha \in \mathcal{C}l(\varphi)$ such that $\alpha \in F$. Similarly, given an atom F and a relation $R \in \{A, B, \bar{B}\}$, we denote by $\mathcal{R}eq_R(F)$ the set of all R-*requests* of F, namely, the formulas $\alpha \in \mathcal{C}l(\varphi)$ such that $\langle R\rangle\alpha \in F$. Taking advantage of the above sets, we can define the following two relations between atoms F and G:

$$F \xrightarrow{A} G \qquad \text{iff} \qquad \mathcal{R}eq_A(F) \;=\; \mathcal{O}bs(G) \cup \mathcal{R}eq_B(G) \cup \mathcal{R}eq_{\bar{B}}(G)$$

$$F \xrightarrow{B} G \qquad \text{iff} \qquad \begin{cases} \mathcal{O}bs(F) \cup \mathcal{R}eq_{\bar{B}}(F) \;\subseteq\; \mathcal{R}eq_{\bar{B}}(G) \;\subseteq\; \mathcal{O}bs(F) \cup \mathcal{R}eq_{\bar{B}}(F) \cup \mathcal{R}eq_B(F) \\ \mathcal{O}bs(G) \cup \mathcal{R}eq_B(G) \;\subseteq\; \mathcal{R}eq_B(F) \;\subseteq\; \mathcal{O}bs(G) \cup \mathcal{R}eq_B(G) \cup \mathcal{R}eq_{\bar{B}}(G). \end{cases}$$

Note that the relation $\xrightarrow{B}$ is transitive, while $\xrightarrow{A}$ is not. Moreover, both $\xrightarrow{A}$ and $\xrightarrow{B}$ satisfy a *view-to-type dependency*, namely, for every pair of intervals I, J in $\mathcal{S}$, we have that

$$\begin{array}{lll} I\ A\ J & \text{implies} & \mathcal{T}ype_{\mathcal{S}}(I) \xrightarrow{A} \mathcal{T}ype_{\mathcal{S}}(J) \\ I\ B\ J & \text{implies} & \mathcal{T}ype_{\mathcal{S}}(I) \xrightarrow{B} \mathcal{T}ype_{\mathcal{S}}(J). \end{array}$$

[2]It is not difficult to show that $\mathsf{AB\bar{B}}$ subsumes the metric extension of A given in [8]. A simple game-theoretic argument shows that the former is in fact strictly more expressive than the latter.

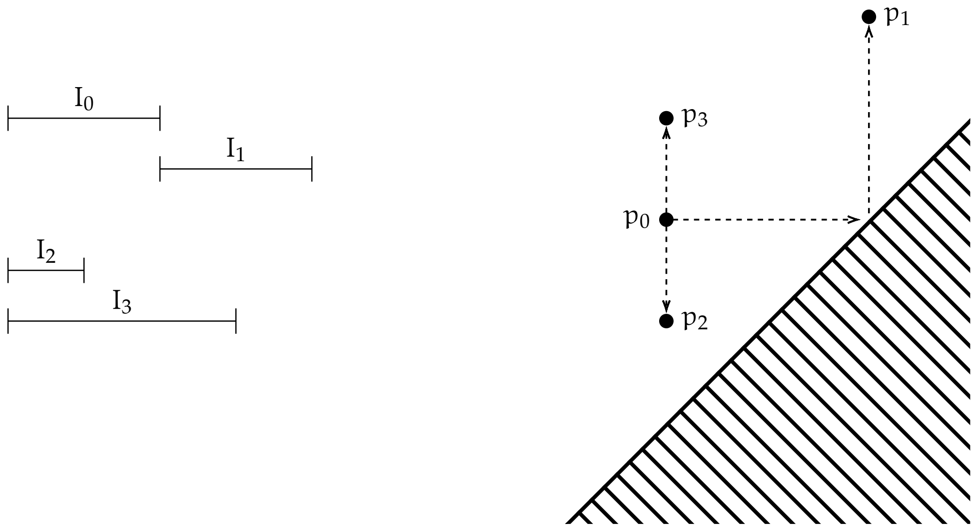

Figure 1: Correspondence between intervals and points of a discrete grid.

Relations $\xrightarrow{A}$ and $\xrightarrow{B}$ will come into play in the definition of consistency conditions (see Definition 2.1).

2.3. Compass structures

The logic $\mathsf{AB\bar{B}}$ can be equivalently interpreted over grid-like structures (the so-called compass structures [18]) by exploiting the existence of a natural bijection between the intervals $I = [x, y]$ and the points $p = (x, y)$ of an $N \times N$ grid such that $x < y$. As an example, Figure 1 depicts four intervals $I_0, ..., I_3$ such that $I_0\ A\ I_1$, $I_0\ B\ I_2$, and $I_0\ \bar{B}\ I_3$, together with the corresponding points $p_0, ..., p_3$ of a discrete grid (note that the three Allen's relations $A, B, \bar{B}$ between intervals are mapped to corresponding spatial relations between points; for the sake of readability, we name the latter ones as the former ones).

Definition 2.1. Given an $\mathsf{AB\bar{B}}$ formula φ, a (consistent and fulfilling) *compass* (φ-)*structure* of length $N \leqslant \omega$ is a pair $\mathcal{G} = (\mathbb{P}_N, \mathcal{L})$, where $\mathbb{P}_N$ is the set of points $p = (x, y)$, with $0 \leqslant x < y < N$, and $\mathcal{L}$ is function that maps any point $p \in \mathbb{P}_N$ to a (φ-)atom $\mathcal{L}(p)$ in such a way that

- for every pair of points $p, q \in \mathbb{P}_N$ and every relation $R \in \{A, B\}$, if $p\ R\ q$ holds, then $\mathcal{L}(p) \xrightarrow{R} \mathcal{L}(q)$ follows (**consistency**);
- for every point $p \in \mathbb{P}_N$, every relation $R \in \{A, B, \bar{B}\}$, and every formula $\alpha \in \mathcal{R}eq_R(\mathcal{L}(p))$, there is a point $q \in \mathbb{P}_N$ such that $p\ R\ q$ and $\alpha \in \mathcal{O}bs(\mathcal{L}(q))$ (**fulfillment**).

We say that a compass (φ-)structure $\mathcal{G} = (\mathbb{P}_N, \mathcal{L})$ *features* a formula α if there is a point $p \in \mathbb{P}_N$ such that $\alpha \in \mathcal{L}(p)$. The following proposition implies that the satisfiability problem for $\mathsf{AB\bar{B}}$ is reducible to the problem of deciding, for any given formula φ, whether there exists a φ-compass structure that features φ.

Proposition 2.2. *An $\mathsf{AB\bar{B}}$-formula φ is satisfied by some interval structure if and only if it is featured by some (φ-)compass structure.*

3. Deciding the satisfiability problem for $\mathsf{AB\bar{B}}$

In this section, we prove that the satisfiability problem for $\mathsf{AB\bar{B}}$ is decidable by providing a "small-model theorem" for the satisfiable formulas of the logic. For the sake of simplicity, we first show that the satisfiability problem for $\mathsf{AB\bar{B}}$ interpreted over *finite* interval structures is decidable and then we generalize such a result to all (finite or infinite) interval structures.

As a preliminary step, we introduce the key notion of shading. Let $\mathcal{G} = (\mathbb{P}_N, \mathcal{L})$ be a compass structure of length $N \leqslant \omega$ and let $0 \leqslant y < N$. The *shading of the row* y *of* $\mathcal{G}$ is the set $\mathcal{S}hading_{\mathcal{G}}(y) = \{\mathcal{L}(x, y) : 0 \leqslant x < y\}$, namely, the set of the atoms of all points in the row y of $\mathcal{G}$ (basically, we interpret different atoms as different colors). Clearly, for every pair of atoms F and F' in $\mathcal{S}hading_{\mathcal{G}}(y)$, we have $\mathcal{R}eq_A(F) = \mathcal{R}eq_A(F')$.

3.1. A small-model theorem for finite structures

Let φ be a generic $\mathsf{AB\bar{B}}$ formula. Let us assume that φ is featured by a *finite* compass structure $\mathcal{G} = (\mathbb{P}_N, \mathcal{L})$, with $N < \omega$. In fact, without loss of generality, we can assume that φ belongs to the atom associated with a point $p = (0, y)$ of $\mathcal{G}$, with $0 < y < N$. We prove that we can restrict our attention to compass structures $\mathcal{G} = (\mathbb{P}_N, \mathcal{L})$, where N is bounded by a double exponential in $|\varphi|$. We start with the following lemma that proves a simple, but crucial, property of the relations $\xrightarrow{A}$ and $\xrightarrow{B}$ (a short proof is given in Section A.1 of the appendix).

Lemma 3.1. *If* $F \xrightarrow{A} H$ *and* $G \xrightarrow{B} H$ *hold for some atoms* F, G, H*, then* $F \xrightarrow{A} G$ *holds as well.*

The next lemma shows that, under suitable conditions, a given compass structure $\mathcal{G}$ may be reduced in length, preserving the existence of atoms featuring φ.

Lemma 3.2. *Let* $\mathcal{G}$ *be a compass structure featuring* φ*. If there exist two rows* $0 < y_0 < y_1 < N$ *in* $\mathcal{G}$ *such that* $\mathcal{S}hading_{\mathcal{G}}(y_0) \subseteq \mathcal{S}hading_{\mathcal{G}}(y_1)$*, then there exists a compass structure* $\mathcal{G}'$ *of length* $N' < N$ *that features* φ*.*

Proof. Suppose that $0 < y_0 < y_1 < N$ are two rows of $\mathcal{G}$ such that $\mathcal{S}hading_{\mathcal{G}}(y_0) \subseteq \mathcal{S}hading_{\mathcal{G}}(y_1)$. Then, there is a function $f : \{0, ..., y_0 - 1\} \to \{0, ..., y_1 - 1\}$ such that, for every $0 \leqslant x < y_0$, $\mathcal{L}(x, y_0) = \mathcal{L}(f(x), y_1)$. Let $k = y_1 - y_0$, $N' = N - k$ ($< N$), and $\mathbb{P}_{N'}$ be the portion of the grid that consists of all points $p = (x, y)$, with $0 \leqslant x < y < N'$. We extend f to a function that maps points in $\mathbb{P}_{N'}$ to points in $\mathbb{P}_N$ as follows:

- if $p = (x, y)$, with $0 \leqslant x < y < y_0$, then we simply let $f(p) = p$;
- if $p = (x, y)$, with $0 \leqslant x < y_0 \leqslant y$, then we let $f(p) = (f(x), y + k)$;
- if $p = (x, y)$, with $y_0 \leqslant x < y$, then we let $f(p) = (x + k, y + k)$.

We denote by $\mathcal{L}'$ the labeling of $\mathbb{P}_{N'}$ such that, for every point $p \in \mathbb{P}_{N'}$, $\mathcal{L}'(p) = \mathcal{L}(f(p))$ and we denote by $\mathcal{G}'$ the resulting structure $(\mathbb{P}_{N'}, \mathcal{L}')$ (see Figure 2). We have to prove that $\mathcal{G}'$ is a *consistent* and *fulfilling* compass structure that features φ (see Definition 2.1). First, we show that $\mathcal{G}'$ satisfies the consistency conditions for the relations B and A; then we show that $\mathcal{G}'$ satisfies the fulfillment conditions for the $\bar{B}$-, B-, and A-requests; finally, we show that $\mathcal{G}'$ features φ.

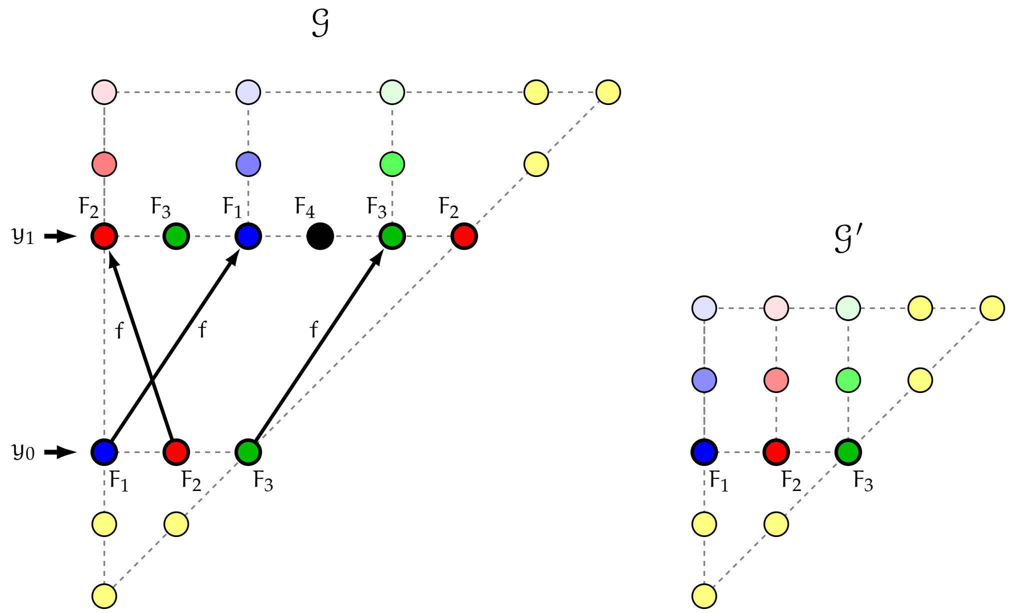

Figure 2: Contraction $\mathcal{G}'$ of a compass structure $\mathcal{G}$.

Consistency with relation B. Consider two points $p = (x, y)$ and $p' = (x', y')$ in $\mathcal{G}'$ such that $p \; B \; p'$, i.e., $0 \leqslant x = x' < y' < y < N'$. We prove that $\mathcal{L}'(p) \xrightarrow{B} \mathcal{L}'(p')$ by distinguishing among the following three cases (note that exactly one of such cases holds):

(1) $y < y_0$ and $y' < y_0$,

(2) $y \geqslant y_0$ and $y' \geqslant y_0$,

(3) $y \geqslant y_0$ and $y' < y_0$.

If $y < y_0$ and $y' < y_0$, then, by construction, we have $f(p) = p$ and $f(p') = p'$. Since $\mathcal{G}$ is a (consistent) compass structure, we immediately obtain $\mathcal{L}'(p) = \mathcal{L}(p) \xrightarrow{B} \mathcal{L}(p') = \mathcal{L}'(p')$.
If $y \geqslant y_0$ and $y \geqslant y_0$, then, by construction, we have either $f(p) = (f(x), y + k)$ or $f(p) = (x + k, y + k)$, depending on whether $x < y_0$ or $x \geqslant y_0$. Similarly, we have either $f(p') = (f(x'), y'+k) = (f(x), y'+k)$ or $f(p') = (x'+k, y'+k) = (x+k, y'+k)$. This implies $f(p) \; B \; f(p')$ and thus, since $\mathcal{G}$ is a (consistent) compass structure, we have $\mathcal{L}'(p) = \mathcal{L}(f(p)) \xrightarrow{B} \mathcal{L}(f(p')) = \mathcal{L}'(p')$.
If $y \geqslant y_0$ and $y' < y_0$, then, since $x < y' < y_0$, we have by construction $f(p) = (f(x), y + k)$ and $f(p') = p'$. Moreover, if we consider the point $p'' = (x, y_0)$ in $\mathcal{G}'$, we easily see that (i) $f(p'') = (f(x), y_1)$, (ii) $f(p) \; B \; f(p'')$ (whence $\mathcal{L}(f(p)) \xrightarrow{B} \mathcal{L}(f(p''))$), (iii) $\mathcal{L}(f(p'')) = \mathcal{L}(p'')$, and (iv) $p'' \; B \; p'$ (whence $\mathcal{L}(p'') \xrightarrow{B} \mathcal{L}(p')$). It thus follows that $\mathcal{L}'(p) = \mathcal{L}(f(p)) \xrightarrow{B} \mathcal{L}(f(p'')) = \mathcal{L}(p'') \xrightarrow{B} \mathcal{L}(p') = \mathcal{L}(f(p')) = \mathcal{L}'(p')$. Finally, by exploiting the transitivity of the relation $\xrightarrow{B}$, we obtain $\mathcal{L}'(p) \xrightarrow{B} \mathcal{L}'(p')$.

Consistency with relation A. Consider two points $p = (x, y)$ and $p' = (x', y')$ such that $p \; A \; p'$, i.e., $0 \leqslant x < y = x' < y' < N'$. We define $p'' = (y, y + 1)$ in such a way that $p \; A \; p''$ and $p' \; B \; p''$ and we distinguish between the following two cases:

(1) $y \geqslant y_0$,

(2) $y < y_0$.

If $y \geqslant y_0$, then, by construction, we have $f(p)\ A\ f(p'')$. Since $\mathcal{G}$ is a (consistent) compass structure, it follows that $\mathcal{L}'(p) = \mathcal{L}(f(p)) \xrightarrow{A} \mathcal{L}(f(p'')) = \mathcal{L}'(p'')$.
If $y < y_0$, then, by construction, we have $\mathcal{L}(p'') = \mathcal{L}(f(p''))$. Again, since $\mathcal{G}$ is a (consistent) compass structure, it follows that $\mathcal{L}'(p) = \mathcal{L}(f(p)) = \mathcal{L}(p) \xrightarrow{A} \mathcal{L}(p'') = \mathcal{L}(f(p'')) = \mathcal{L}'(p'')$.
In both cases we have $\mathcal{L}'(p) \xrightarrow{A} \mathcal{L}'(p'')$. Now, we recall that $p'\ B\ p''$ and that, by previous arguments, $\mathcal{G}'$ is consistent with the relation B. We thus have $\mathcal{L}'(p') \xrightarrow{B} \mathcal{L}'(p'')$. Finally, by applying Lemma 3.1, we obtain $\mathcal{L}'(p) \xrightarrow{A} \mathcal{L}'(p')$.

Fulfillment of B-requests. Consider a point $p = (x, y)$ in $\mathcal{G}'$ and some B-request $\alpha \in \mathcal{R}eq_B\big(\mathcal{L}'(p)\big)$ associated with it. Since, by construction, $\alpha \in \mathcal{R}eq_B\big(\mathcal{L}(f(p))\big)$ and $\mathcal{G}$ is a (fulfilling) compass structure, we know that $\mathcal{G}$ contains a point $q' = (x', y')$ such that $f(p)\ B\ q'$ and $\alpha \in \mathcal{O}bs\big(\mathcal{L}(q')\big)$. We prove that $\mathcal{G}'$ contains a point p' such that $p\ B\ p'$ and $\alpha \in \mathcal{O}bs\big(\mathcal{L}'(p')\big)$ by distinguishing among the following three cases (note that exactly one of such cases holds):

(1) $y < y_0$

(2) $y' \geqslant y_1$,

(3) $y \geqslant y_0$ and $y' < y_1$.

If $y < y_0$, then, by construction, we have $p = f(p)$ and $q' = f(q')$. Therefore, we simply define $p' = q'$ in such a way that $p = f(p)\ B\ q' = p'$ and $\alpha \in \mathcal{O}bs\big(\mathcal{L}'(p')\big)$ $\big(= \mathcal{O}bs\big(\mathcal{L}(f(p'))\big) = \mathcal{O}bs\big(\mathcal{L}(q')\big)\big)$.
If $y' \geqslant y_1$, then, by construction, we have either $f(p) = (f(x), y+k)$ or $f(p) = (x+k, y+k)$, depending on whether $x < y_0$ or $x \geqslant y_0$. We define $p' = (x, y' - k)$ in such a way that $p\ B\ p'$. Moreover, we observe that either $f(p') = (f(x), y')$ or $f(p') = (x+k, y')$, depending on whether $x < y_0$ or $x \geqslant y_0$, and in both cases $f(p') = q'$ follows. This shows that $\alpha \in \mathcal{O}bs\big(\mathcal{L}'(p')\big)$ $\big(= \mathcal{O}bs\big(\mathcal{L}(f(p')\big) = \mathcal{O}bs\big(\mathcal{L}(q')\big)\big)$.
If $y \geqslant y_0$ and $y' < y_1$, then we define $\bar{p} = (x, y_0)$ and $\bar{q} = (x', y_1)$ and we observe that $f(p)\ B\ \bar{q}\ B\ q'$ and $f(\bar{p}) = \bar{q}$. Since $f(p)\ B\ \bar{q})$, we have that $\alpha \in \mathcal{R}eq_B\big(\mathcal{L}(\bar{q})\big)$ and hence $\alpha \in \mathcal{R}eq_B\big(\mathcal{L}(\bar{p})\big)$. Since $\mathcal{G}$ is a (fulfilling) compass structure, we know that there is a point p' such that $\bar{p}\ B\ p'$ and $\alpha \in \mathcal{O}bs\big(\mathcal{L}(\bar{p}')\big)$. Moreover, since $\bar{p}\ B\ p'$, we have $f(p') = p'$, from which we obtain $p\ B\ p'$ and $\alpha \in \mathcal{O}bs\big(\mathcal{L}(p')\big)$.

Fulfillment of $\bar{B}$-requests. The proof that $\mathcal{G}'$ fulfills all $\bar{B}$-requests of its atoms is symmetric with respect to the previous one.

Fulfillment of A-requests. Consider a point $p = (x, y)$ in $\mathcal{G}'$ and some A-request $\alpha \in \mathcal{R}eq_A\big(\mathcal{L}'(p)\big)$ associated with p in $\mathcal{G}'$. Since, by previous arguments, $\mathcal{G}'$ fulfills all $\bar{B}$-requests of its atoms, it is sufficient to prove that either $\alpha \in \mathcal{O}bs\big(\mathcal{L}'(p')\big)$ or $\alpha \in \mathcal{R}eq_{\bar{B}}\big(\mathcal{L}'(p')\big)$, where $p' = (y, y+1)$. This can be easily proved by distinguishing among the three cases $y < y_0 - 1$, $y = y_0 - 1$, and $y \geqslant y_0$.

Featured formulas. Recall that, by previous assumptions, $\mathcal{G}$ contains a point $p = (0, y)$, with $0 < y < N$, such that $\varphi \in \mathcal{L}(p)$. If $y \leqslant y_0$, then, by construction, we have $\varphi \in \mathcal{L}'(p)$ $(= \mathcal{L}(f(p)) = \mathcal{L}(p))$. Otherwise, if $y > y_0$, we define $q = (0, y_0)$ and we observe that $q\ \bar{B}\ p$. Since $\mathcal{G}$ is a (consistent) compass structure and $\langle\bar{B}\rangle\varphi \in \mathcal{C}l^+(\varphi)$, we have that $\varphi \in \mathcal{R}eq_{\bar{B}}\big(\mathcal{L}(q)\big)$. Moreover, by construction, we have $\mathcal{L}'(q) = \mathcal{L}(f(q))$ and hence $\varphi \in \mathcal{R}eq_{\bar{B}}\big(\mathcal{L}'(q)\big)$. Finally, since $\mathcal{G}'$ is a (fulfilling) compass structure, we know that there is a point p' in $\mathcal{G}'$ such that $f(q)\ \bar{B}\ p'$ and $\varphi \in \mathcal{O}bs\big(\mathcal{L}'(p')\big)$. $\square$

On the grounds of the above result, we can provide a suitable upper bound for the length of a minimal finite interval structure that satisfies φ, if there exists any. This yields a straightforward, but inefficient, 2EXPSPACE algorithm that decides whether a given $AB\bar{B}$-formula φ is satisfiable over finite interval structures.

Theorem 3.3. *An $AB\bar{B}$-formula φ is a satisfied by some finite interval structure iff it is featured by some compass structure of length $N \leqslant 2^{2^{7|\varphi|}}$ (i.e., double exponential in $|\varphi|$).*

Proof. One direction is trivial. We prove the other one ("only if" part). Suppose that φ is satisfied by a finite interval structure $\mathcal{S}$. By Proposition 2.2, there is a compass structure $\mathcal{G}$ that features φ and has finite length $N < \omega$. Without loss of generality, we can assume that N is minimal among all finite compass structures that feature φ. We recall from Section 2.2 that $\mathcal{G}$ contains at most $2^{7|\varphi|}$ distinct atoms. This implies that there exist at most $2^{2^{7|\varphi|}}$ different shadings of the form $\mathcal{S}hading_{\mathcal{G}}(y)$, with $0 \leqslant y < N$. Finally, by applying Lemma 3.2, we obtain $N \leqslant 2^{2^{7|\varphi|}}$ (otherwise, there would exist two rows $0 < y_0 < y_1 < N$ such that $\mathcal{S}hading_{\mathcal{G}}(y_0) = \mathcal{S}hading_{\mathcal{G}}(y_1)$, which is against the hypothesis of minimality of N). ☐

3.2. A small-model theorem for infinite structures

In general, compass structures that feature φ may be infinite. Here, we prove that, without loss of generality, we can restrict our attention to sufficiently "regular" infinite compass structures, which can be represented in double exponential space with respect to $|\varphi|$. To do that, we introduce the notion of periodic compass structure.

Definition 3.4. An infinite compass structure $\mathcal{G} = (\mathbb{P}_\omega, \mathcal{L})$ is *periodic*, with *threshold* $\widetilde{y}_0$, *period* $\widetilde{y}$, and *binding* $\widetilde{g} : \{0, ..., \widetilde{y}_0 + \widetilde{y} - 1\} \to \{0, ..., \widetilde{y}_0 - 1\}$, if the following conditions are satisfied:

- for every $\widetilde{y}_0 + \widetilde{y} \leqslant x < y$, we have $\mathcal{L}(x, y) = \mathcal{L}(x - \widetilde{y}, y - \widetilde{y})$,
- for every $0 \leqslant x < \widetilde{y}_0 + \widetilde{y} \leqslant y$, we have $\mathcal{L}(x, y) = \mathcal{L}(\widetilde{g}(x), y - \widetilde{y})$.

Figure 3 gives an example of a periodic compass structure (the arrows represent some relationships between points induced by the binding function $\widetilde{g}$). Note that any periodic compass structure $\mathcal{G} = (\mathbb{P}_\omega, \mathcal{L})$ can be finitely represented by specifying (i) its threshold $\widetilde{y}_0$, (ii) its period $\widetilde{y}$, (iii) its binding $\widetilde{g}$, and (iv) the labeling $\mathcal{L}$ restricted to the portion $\mathbb{P}_{\widetilde{y}_0 + \widetilde{y} - 1}$ of the domain.

The following theorem leads immediately to a 2EXPSPACE algorithm that decides whether a given $AB\bar{B}$-formula φ is satisfiable over infinite interval structures (its proof is given in Section A.2 of the appendix).

Theorem 3.5. *An $AB\bar{B}$-formula φ is satisfied by an infinite interval structure iff it is featured by a periodic compass structure with threshold $\widetilde{y}_0 < 2^{2^{7|\varphi|}}$ and period $\widetilde{y} < 2|\varphi| \cdot 2^{2^{7|\varphi|}} \cdot 2^{2^{7|\varphi|}}$.*

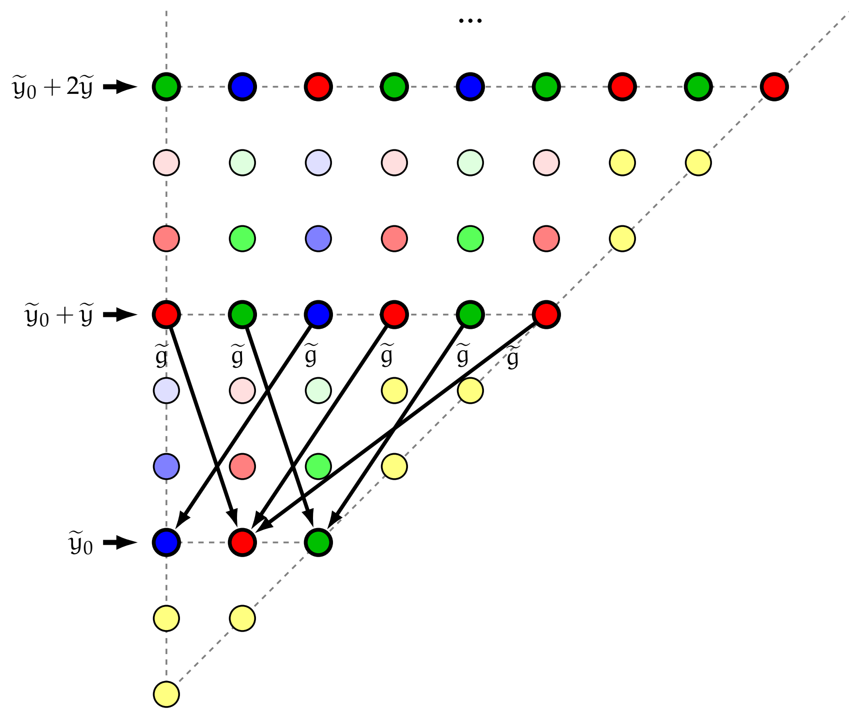

Figure 3: A periodic compass structure with threshold $\widetilde{y}_0$, period $\widetilde{y}$, and binding $\widetilde{g}$.

4. Tight complexity bounds to the satisfiability problem for $\mathsf{AB\bar{B}}$

In this section, we show that the satisfiability problem for $\mathsf{AB\bar{B}}$ interpreted over (either finite or infinite) interval temporal structures is EXPSPACE-complete.

The EXPSPACE-hardness of the satisfiability problem for $\mathsf{AB\bar{B}}$ follows from a reduction from the *exponential-corridor tiling problem*, which is known to be EXPSPACE-complete [17]. Formally, an instance of the exponential-corridor tiling problem is a tuple $\mathcal{T} = (T, t_\perp, t_\top, H, V, n)$ consisting of a finite set T of tiles, a bottom tile $t_\perp \in T$, a top tile $t_\top \in T$, two binary relations H, V over T (specifying the horizontal and vertical constraints), and a positive natural number n (represented in unary notation). The problem consists in deciding whether there exists a tiling $f : \mathbb{N} \times \{0, ..., 2^n - 1\} \rightarrow T$ of the infinite discrete corridor of height 2^n, that associates the tile $t_\perp$ (resp., $t_\top$) with the bottom (resp., top) row of the corridor and that respects the horizontal and vertical constraints H and V, namely,

i) for every $x \in \mathbb{N}$, we have $f(x, 0) = t_\perp$,

ii) for every $x \in \mathbb{N}$, we have $f(x, 2^n - 1) = t_\top$,

iii) for every $x \in \mathbb{N}$ and every $0 \leqslant y < 2^n$, we have $f(x, y)\ H\ f(x + 1, y)$,

iv) for every $x \in \mathbb{N}$ and every $0 \leqslant y < 2^n - 1$, we have $f(x, y)\ V\ f(x, y + 1)$.

The proof of the following lemma, which reduces the exponential-corridor tiling problem to the satisfiability problem for $\mathsf{AB\bar{B}}$, is given in Section A.3 of the appendix. Intuitively, such a reduction exploits (i) the correspondence between the points $p = (x, y)$ inside the infinite corridor $\mathbb{N} \times \{0, ..., 2^n - 1\}$ and the intervals of the form $I_p = [y + 2^n x, y + 2^n x + 1]$, (ii) $|T|$ propositional variables which represent the tiling function f, (iii) n additional propositional variables which represent (the binary expansion of) the y-coordinate of each row of the corridor, and (iv) the modal operators $\langle \mathsf{A} \rangle$ and $\langle \mathsf{B} \rangle$ by means of which one can enforce the local constrains over the tiling function f (as a matter of fact, this shows that the satisfiability problem for the AB fragment is already hard for EXPSPACE).

Lemma 4.1. *There is a polynomial-time reduction from the exponential-corridor tiling problem to the satisfiability problem for* $\mathsf{AB\bar{B}}$.

As for the EXPSPACE-completeness, we claim that the existence of a compass structure $\mathcal{G}$ that features a given formula φ can be decided by verifying suitable local (and stronger) consistency conditions over all pairs of contiguous rows. In fact, in order to check that these local conditions hold between two contiguous rows y and $y+1$, it is sufficient to store into memory a bounded amount of information, namely, (i) a counter y that ranges over $\{1, ..., 2^{2^{7|\varphi|}} + |\varphi| \cdot 2^{2^{7|\varphi|}}\}$, (ii) the two guessed shadings S and S' associated with the rows y and $y+1$, and (iii) a function $g : \mathsf{S} \rightarrow \mathsf{S}'$ that captures the horizontal alignment relation between points with an associated atom from S and points with an associated atom from S'. This shows that the satisfiability problem for $\mathsf{AB\bar{B}}$ can be decided in exponential space, as claimed by the following lemma. Further details about the decision procedure, including soundness and completeness proofs, can be found in Section A.4 of the appendix.

Lemma 4.2. *There is an EXPSPACE non-deterministic procedure that decides whether a given formula of* $\mathsf{AB\bar{B}}$ *is satisfiable or not.*

Summing up, we obtain the following tight complexity result.

Theorem 4.3. *The satisfiability problem for* $\mathsf{AB\bar{B}}$ *interpreted over (prefixes of) natural numbers is EXPSPACE-complete.*

5. Conclusions

In this paper, we proved that the satisfiability problem for $\mathsf{AB\bar{B}}$ interpreted over prefixes of the natural numbers is EXPSPACE-complete. We restricted our attention to these domains because it is a common commitment in computer science. Moreover, this gave us the possibility of expressing meaningful metric constraints in a fairly natural way. Nevertheless, we believe it possible to extend our results to the class of all linear orderings as well as to relevant subclasses of it. Another restriction that can be relaxed is the one about singleton intervals: all results in the paper can be easily generalized to include singleton intervals in the underlying structure $\mathbb{I}_{\mathbb{N}}$. The most exciting challenge is to establish whether the modality $\bar{\mathsf{A}}$ can be added to $\mathsf{AB\bar{B}}$ preserving decidability (note that $\langle \mathsf{A} \rangle$, $\langle \mathsf{B} \rangle$, and $\langle \bar{\mathsf{B}} \rangle$ are all future modalities, while $\langle \bar{\mathsf{A}} \rangle$ is a past one). Preliminary results seem to suggest that the addition $\langle \bar{\mathsf{A}} \rangle$ involves a non-elementary blow-up in computational complexity, but it does not destroy decidability.

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Appendix A. Appendix

In this appendix, we report some complete proofs that have been omitted in the previous sections. Moreover, we describe an EXPSPACE (optimal) procedure that decides satisfiability of $\mathsf{AB\bar{B}}$.

A.1. Proof of Lemma 3.1

Lemma 3.1. *If* $\mathsf{F} \xrightarrow{\mathsf{A}} \mathsf{H}$ *and* $\mathsf{G} \xrightarrow{\mathsf{B}} \mathsf{H}$ *hold for some atoms* $\mathsf{F}, \mathsf{G}, \mathsf{H}$*, then* $\mathsf{F} \xrightarrow{\mathsf{A}} \mathsf{G}$ *holds as well.*

Proof. Suppose that $\mathsf{F} \xrightarrow{\mathsf{A}} \mathsf{H}$ and $\mathsf{G} \xrightarrow{\mathsf{B}} \mathsf{H}$ hold for some atoms $\mathsf{F}, \mathsf{G}, \mathsf{H}$. By applying the definitions of the relations $\xrightarrow{\mathsf{A}}$ and $\xrightarrow{\mathsf{B}}$, we immediately obtain:

$$\begin{aligned} \mathcal{R}eq_{\mathsf{A}}(\mathsf{F}) &= \mathcal{O}bs(\mathsf{H}) \cup \mathcal{R}eq_{\mathsf{B}}(\mathsf{H}) \cup \mathcal{R}eq_{\bar{\mathsf{B}}}(\mathsf{H}) && (\text{since } \mathsf{F} \xrightarrow{\mathsf{A}} \mathsf{H}) \\ &= \mathcal{O}bs(\mathsf{G}) \cup \mathcal{R}eq_{\mathsf{B}}(\mathsf{G}) \cup \mathcal{R}eq_{\bar{\mathsf{B}}}(\mathsf{G}) && (\text{since } \mathsf{G} \xrightarrow{\mathsf{B}} \mathsf{H}). \end{aligned}$$

This shows that $\mathsf{F} \xrightarrow{\mathsf{A}} \mathsf{G}$. □

A.2. Proof of Theorem 3.5

Theorem 3.5. *An* $\mathsf{AB\bar{B}}$*-formula* φ *is a satisfied by an infinite interval structure iff it is featured by a periodic compass structure with threshold* $\tilde{y}_0 < 2^{2^{7|\varphi|}}$ *and period* $\tilde{y} < 2|\varphi| \cdot 2^{2^{7|\varphi|}} \cdot 2^{2^{7|\varphi|}}$.

Proof. One direction is trivial. We prove the other one ("only if" part). Suppose that φ is satisfied by an infinite interval structure $\mathcal{S}$. By Proposition 2.2, there is an infinite compass structure $\mathcal{G}$ that features φ. Below, we show how to turn $\mathcal{G}$ into a periodic compass structure $\mathcal{G}'$ that still features φ and whose threshold and period satisfy the bounds given by the theorem.

Threshold $\tilde{y}_0$. Since $\mathcal{G}$ is infinite, we know that there exist infinitely many rows $y_0, y_1, y_2, \ldots$ such that $\mathcal{S}hading_{\mathcal{G}}(y_i) = \mathcal{S}hading_{\mathcal{G}}(y_j)$ for every pair of indices $i, j \in \mathbb{N}$. We define $\tilde{y}_0$ as the least of all such rows. By simple counting arguments, we have that $\tilde{y}_0 < 2^{2^{7|\varphi|}}$.

Period $\tilde{y}$. Since $\mathcal{G}$ is a (fulfilling) compass structure, there is a function f that maps any point $p = (x, \tilde{y}_0)$, any relation $\mathsf{R} \in \{\mathsf{A}, \bar{\mathsf{B}}\}$, and any request $\alpha \in \mathcal{R}eq_{\mathsf{R}}\big(\mathcal{L}(p)\big)$ to a point $p' = f(p, \mathsf{R}, \alpha)$ such that $p \ \mathsf{R} \ p'$ and $\alpha \in \mathcal{O}bs\big(\mathcal{L}(p')\big)$. Let f be one such function. We denote by $\mathcal{I}mg(f)$ the image set of f, namely, the set of all points of the form $p' = f(p, \mathsf{R}, \alpha)$, with $p = (x, \tilde{y}_0)$, $\mathsf{R} \in \{\mathsf{A}, \bar{\mathsf{B}}\}$, and $\alpha \in \mathcal{R}eq_{\mathsf{R}}\big(\mathcal{L}(p)\big)$. Moreover, we denote by $\mathcal{I}mg_y(f)$ the projection of $\mathcal{I}mg(f)$ on the y-component. Intuitively, $\mathcal{I}mg_y(f)$ is a minimal set of rows that fulfill all A-requests and all $\bar{\mathsf{B}}$-requests of atoms along the row $\tilde{y}_0$ in $\mathcal{G}$ (see, for instance, Figure 4). Clearly, $\min\big(\mathcal{I}mg_y(f)\big) > \tilde{y}_0$ and $\mathcal{I}mg_y(f)$ contains at most $2|\varphi| \cdot \tilde{y}_0$ (possibly non-contiguous) rows (namely, at most one row for each choice of $0 \leqslant x < \tilde{y}_0$, $\mathsf{R} \in \{\mathsf{A}, \bar{\mathsf{B}}\}$, and $\alpha \in \mathcal{R}eq_{\mathsf{R}}\big(\mathcal{L}(x, \tilde{y}_0)\big)\big)$. We call *gap* of $\mathcal{I}mg_y(f)$ any set $Y = \{y, y+1, \ldots, y'\}$ of contiguous rows of $\mathcal{G}$ such that $\tilde{y}_0 < y \leqslant y' < \max\big(\mathcal{I}mg_y(f)\big)$ and $\mathcal{I}mg_y(f) \cap Y = \emptyset$. From previous results (in particular, from the proofs of Lemma 3.2 and Theorem 3.3), we can assume, without loss of generality, that every gap Y of $\mathcal{I}mg_y(f)$ has size at most $2^{2^{7|\varphi|}} - 1$ (otherwise,

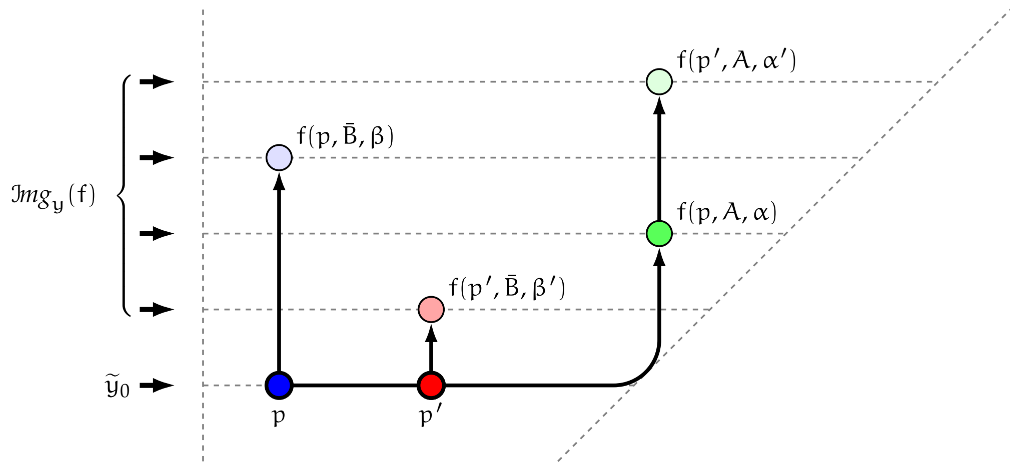

Figure 4: A set $\mathcal{I}mg_y(f)$ of rows that fulfill all requests at row $\widetilde{y}_0$.

we can find two rows y'_0 and y'_1 in Y that satisfy the hypothesis of Lemma 3.2 and hence we can "remove" the rows from y'_0 to $y'_1 - 1$ from $\mathcal{G}$, without affecting consistency and fulfillment). This shows that $\max\big(\mathcal{I}mg_y(f)\big) \leqslant \widetilde{y}_0 + 2|\varphi| \cdot \widetilde{y}_0 \cdot 2^{2^{7|\varphi|}}$. We then define $\widetilde{y}$ as the least value such that $\widetilde{y}_0\widetilde{y} > \max\big(\mathcal{I}mg_y(f)\big)$ and $\mathcal{S}hading_{\mathcal{G}}(\widetilde{y}_0) = \mathcal{S}hading_{\mathcal{G}}(\widetilde{y}_0 + \widetilde{y})$. Again, by exploiting simple counting arguments, one can prove that $\widetilde{y} < \max\big(\mathcal{I}mg_y(f)\big) - \widetilde{y}_0 + 2^{2^{7|\varphi|}}$ $\leqslant 2|\varphi| \cdot \widetilde{y}_0 \cdot 2^{2^{7|\varphi|}} + 2^{2^{7|\varphi|}} \leqslant \big(2|\varphi| \cdot \widetilde{y}_0 + 1\big) \cdot 2^{2^{7|\varphi|}} \leqslant 2|\varphi| \cdot \big(\widetilde{y}_0 + 1\big) \cdot 2^{2^{7|\varphi|}} \leqslant 2|\varphi| \cdot 2^{2^{7|\varphi|}} \cdot 2^{2^{7|\varphi|}}$.

Binding $\widetilde{g}$. Since $\mathcal{S}hading_{\mathcal{G}}(\widetilde{y}_0) = \mathcal{S}hading_{\mathcal{G}}(\widetilde{y}_0 + \widetilde{y})$, we know that there is a (surjective) function g that maps any value $x \in \{0, ..., \widetilde{y}_0 + \widetilde{y} - 1\}$ to a value $g(x) \in \{0, ..., \widetilde{y}_0 - 1\}$ in such a way that $\mathcal{L}(x, \widetilde{y}_0 + \widetilde{y}) = \mathcal{L}(g(x), \widetilde{y}_0)$. We choose one such function as $\widetilde{g}$.

Periodic compass structure $\mathcal{G}'$. According to Definition 3.4, the threshold $\widetilde{y}_0$, the period $\widetilde{y}$, the binding $\widetilde{g}$, and the labeling $\mathcal{L}$ of $\mathcal{G}$ restricted to the finite domain $\mathbb{P}_{\widetilde{y}_0 + \widetilde{y} - 1}$ uniquely determine a periodic structure $\mathcal{G}' = (\mathbb{P}_\omega, \mathcal{L}')$. It thus remains to show that $\mathcal{G}'$ is a (consistent and fulfilling) compass structure that features φ. The proof that the labeling $\mathcal{L}'$ is consistent with the relations A, B, and $\bar{B}$ is straightforward, given the above construction. As for the fulfillment of the various requests, one can prove, by induction on n, that, for every $n \in \mathbb{N}$, every point $p = (x, y)$ with $y = \widetilde{y}_0 + n\widetilde{y}$, every relation $R \in \{A, \bar{B}\}$ (resp., $R = B$), and every R-request $\alpha \in \mathcal{R}eq_R\big(\mathcal{L}'(p)\big)$, there is a point $p' = (x', y')$ such that $y' \leqslant \widetilde{y}_0 + (n+1)\widetilde{y}$ (resp., $y' < \widetilde{y}_0 + n\widetilde{y}$), $p \ R \ p'$, and $\alpha \in \mathcal{O}bs\big(\mathcal{L}'(p')\big)$. This suffices to claim that $\mathcal{G}'$ is a consistent and fulfilling compass structure. Consider the case of relation $\bar{B}$ (the case of relation B is fully symmetric and the case of relation A can be easily reduced to that of $\bar{B}$). By contradiction, let us suppose that there is a point $p = (x, y)$, with $\widetilde{y}_0 + n\widetilde{y} < y < \widetilde{y}_0 + (n+1)\widetilde{y}$, such that $\alpha \in \mathcal{R}eq_{\bar{B}}\big(\mathcal{L}(p)\big)$ and $\alpha \notin \mathcal{O}bs\big(\mathcal{L}(p')\big)$ for all points p' such that $p \ \bar{B} \ p'$. Since $\mathcal{G}'$ is consistent, we have $\alpha \in \mathcal{R}eq_{\bar{B}}\big(\mathcal{L}(q)\big)$, where $q = \big(x, \widetilde{y}_0 + (n+1)\widetilde{y}\big)$ (note that $p \ \bar{B} \ q$ holds) and thus, by construction, there is a point $q' = (x, y')$, with $\widetilde{y}_0 + (n+1)\widetilde{y} < y' \leqslant \widetilde{y}_0 + (n+2)\widetilde{y}$, such that $\alpha \in \mathcal{O}bs\big(\mathcal{L}(q')\big)$ (a contradiction). Finally, one can show that $\mathcal{G}'$ features the formula φ by exploiting the same argument that was given in the proof of Lemma 3.2. $\square$

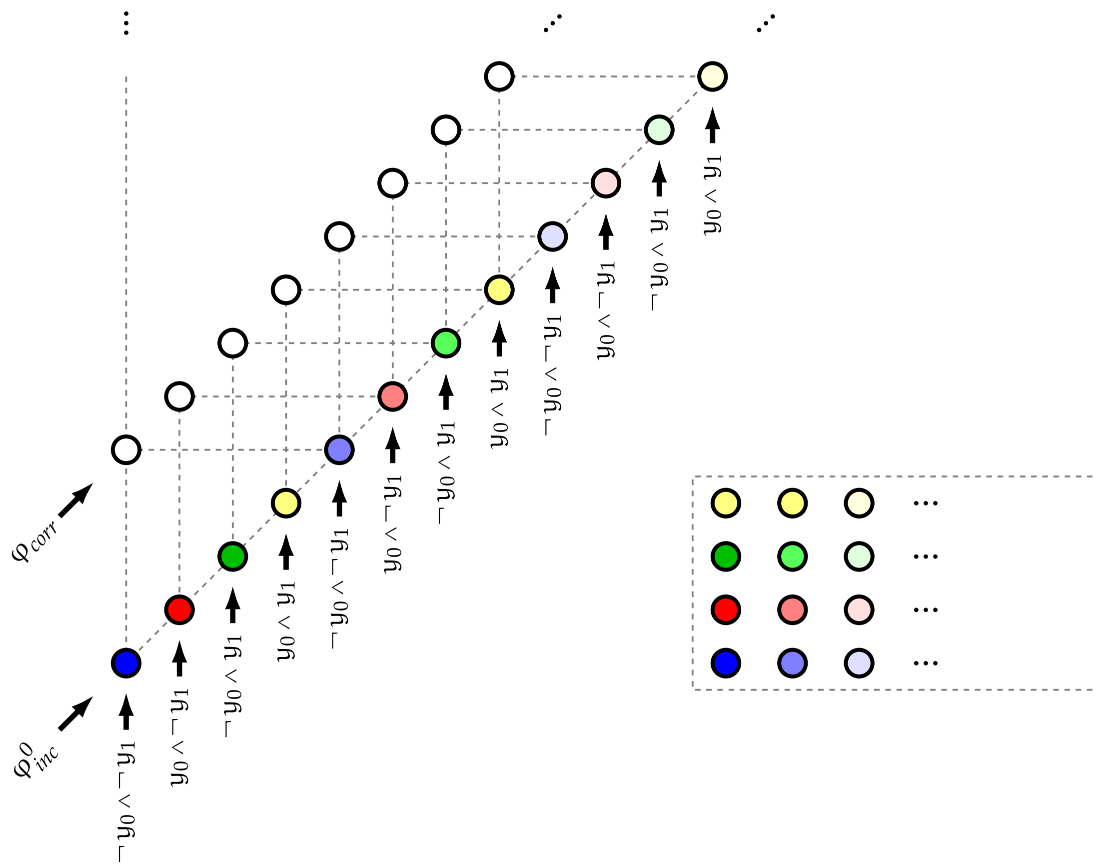

Figure 5: Encoding of a tiling function.

A.3. Proof of Lemma 4.1

Lemma 4.1. *There is a polynomial-time reduction from the exponential-corridor tiling problem to the satisfiability problem for* $AB\bar{B}$.

Proof. Consider a generic instance $\mathcal{T} = (T, t_\perp, t_\top, H, V, n)$ of the exponential-corridor tiling problem, where $T = \{t_1, \ldots, t_k\}$. We guarantee the existence of a tiling function $f : \mathbb{N} \times \{0, ..., 2^n - 1\} \rightarrow T$ that satisfies the instance $\mathcal{T}$ through the existence of a labeled (infinite) interval structure $\mathcal{S} = (\mathbb{I}_\omega, A, B, \sigma)$ that satisfies a suitable AB formula with size polynomial in $\mathcal{T}$. We use k propositional variables $t_1, ..., t_k$ to represent the tiles from T, n propositional variables $y_0, ..., y_{n-1}$ to represent the binary expansion of the y-coordinate of a row, and one propositional variable c to identify those intervals in $\mathbb{I}_\omega$ that correspond to points of the infinite corridor of height 2^n. The correspondence between the points $p = (x, y)$, with $x \in \mathbb{N}$ and $0 \leqslant y < 2^n$, of the infinite corridor and the intervals $I_p \in \mathbb{I}_\omega$ is obtained by letting $I_p = [y + 2^n x, y + 2^n x + 1]$ (Figure 5 can be used as a reference example through the rest of the proof). According to such an encoding, the labeling function σ is related to the tiling function f as follows:

for every point $p = [x, y] \in \mathbb{N} \times \{0, ..., 2^n - 1\}$ *and every index* $1 \leqslant i \leqslant k$*, if* $f(p) = t_i$*, then* $\sigma(I_p) = \{c, t_i, y_{j_1}, ..., y_{j_h}\}$*, where* $\{j_1, ..., j_j\} \subseteq \{0, ..., n-1\}$ *and* $y = \sum_{j \in \{j_1,...,j_h\}} 2^j$.

For the sake of brevity, we introduce a universal modal operator $[U]$, which is defined as follows:

$$[U]\alpha \;=\; \alpha \;\wedge\; [A]\alpha \;\wedge\; [A][A]\alpha.$$

We now show how to express the existence of a tiling function f that satisfies $\mathcal{T}$. First of all, we associate the propositional variable c with all and only the intervals of the form $I_p = [y + 2^n x, y + 2^n x + 1]$, with $x \in \mathbb{N}$ and $0 \leqslant y < 2^n$ (atomic intervals), as follows:

$$\varphi_c \;=\; [U](c \;\leftrightarrow\; [B]\bot).$$

The tiling function $f : \mathbb{N} \times \{0, ..., 2^n - 1\} \;\to\; T$ is represented by associating with each c-labeled interval $I_p = [y + 2^n x, y + 2^n x + 1]$ a unique propositional variable $f(p)$ in T as follows:

$$\varphi_f \;=\; [U]\Big(c \;\to\; \bigvee\nolimits_{1\leqslant i\leqslant k} t_i\Big) \;\wedge\; [U]\Big(c \;\to\; \bigwedge\nolimits_{1\leqslant i<j\leqslant k} \neg(t_i \;\wedge\; t_j)\Big).$$

Next, we associate with each (possibly non-minimal) interval of the form $I = [y + 2^n x, y + 2^n x + l]$ a subset of the propositional variables $y_0, ..., y_{n-1}$ that encodes the binary expansion of y. Such a labeling can be enforced by the formula:

$$\varphi_y \;=\; \Big(\bigwedge\nolimits_{0\leqslant i<n} \neg y_i\Big) \;\wedge\; [U]\Big(\bigwedge\nolimits_{0\leqslant i<n} (y_i \leftrightarrow [B]y_i) \wedge \big(\neg y_i \leftrightarrow [B]\neg y_i\big)\Big) \;\wedge\; [U]\Big(c \to \varphi^0_{inc}\Big)$$

where the formula φ^i_{inc} is defined (by induction on $i \in \{n, ..., 0\}$) as follows:

$$\varphi^i_{inc} \;=\; \begin{cases} \top & \text{if } i = n, \\ \Big(y_i \wedge \langle A\rangle(c \wedge \neg y_i) \wedge \varphi^{i+1}_{inc}\Big) \;\vee\; \Big(\neg y_i \wedge \langle A\rangle(c \wedge y_i) \wedge \varphi^{i+1}_{eq}\Big) & \text{if } i < n, \end{cases}$$

The formula φ^i_{inc} involves the formula φ^i_{eq}, which is defined (by induction on $i \in \{n, ..., 0\}$) as follows:

$$\varphi^i_{eq} \;=\; \begin{cases} \top & \text{if } i = n, \\ \Big(\big(y_i \;\wedge\; \langle A\rangle(c \wedge y_i)\big) \;\vee\; \big(\neg y_i \;\wedge\; \langle A\rangle(c \wedge \neg y_i)\Big) \;\wedge\; \varphi^{i+1}_{eq} & \text{if } i < n. \end{cases}$$

It remains to express the constraints on the tiling function f. This can be done by using the following formulas (for the sake of simplicity, we assume, without loss of generality, that $(t_\top, t_\bot) \in V$):

$$\varphi_\bot \;=\; [U]\Big(c \;\wedge\; \bigwedge\nolimits_{0\leqslant i<n} \neg y_i \;\;\to\;\; t_\bot\Big)$$

$$\varphi_\top \;=\; [U]\Big(c \;\wedge\; \bigwedge\nolimits_{0\leqslant i<n} y_i \;\;\to\;\; t_\top\Big)$$

$$\varphi_H \;=\; [U]\bigwedge\nolimits_{1\leqslant i\leqslant k}\Big(\big(\varphi_{corr} \;\wedge\; \langle B\rangle(c \;\wedge\; t_i)\big) \;\to\; \bigvee\nolimits_{(t_i,t_j)\in H} \langle A\rangle(c \;\wedge\; t_j)\Big)$$

$$\varphi_V \;=\; [U]\bigwedge\nolimits_{1\leqslant i\leqslant k}\Big((c \;\wedge\; t_i) \;\to\; \bigvee\nolimits_{(t_i,t_j)\in V} \langle A\rangle(c \;\wedge\; t_j)\Big),$$

where $\varphi_{corr} = \varphi^0_{eq} \wedge [\mathsf{B}]\neg\varphi^0_{eq}$ (intuitively, the formula φ_{corr} holds over all and only the intervals of the form $\mathrm{I} = \big[y + 2^n x, y + 2^n(x+1)\big]$, in such a way that, if J and K are the shortest intervals such that I B J and I A K, then J corresponds to the point $p = (x, y)$ and K corresponds to the point $q = (x+1, y)$).

Summing up, we have that the formula $\varphi = \varphi_c \wedge \varphi_f \wedge \varphi_y \wedge \varphi_\perp \wedge \varphi_\top \wedge \varphi_H \wedge \varphi_V$, which has polynomial size in $|\mathcal{T}|$ and uses only the modal operators $\langle \mathsf{A}\rangle$ and $\langle \mathsf{B}\rangle$, is satisfiable if and only if $\mathcal{T}$ is a positive instance of the exponential-corridor tiling problem. $\square$

A.4. Proof of Lemma 4.2

Lemma 4.2. *There is an EXPSPACE non-deterministic procedure that decides whether a given formula of* $\mathsf{AB\bar{B}}$ *is satisfiable or not.*

In order to prove this lemma, we need to introduce two variants of the dependency relations $\xrightarrow{\mathsf{B}}$ and $\xrightarrow{\mathsf{A}}$, which are more restrictive than the previous ones and which are evaluated (locally) over pairs of atoms that lie along two contiguous rows. Precisely, we define the following relations between atoms F and G:

$$\mathsf{F} \xmapsto{\mathsf{A}} \mathsf{G} \qquad \text{iff} \qquad \begin{cases} \mathcal{R}eq_{\mathsf{A}}(\mathsf{F}) \;=\; \mathcal{O}bs(\mathsf{G}) \cup \mathcal{R}eq_{\bar{\mathsf{B}}}(\mathsf{G}) \\ \mathcal{R}eq_{\mathsf{B}}(\mathsf{G}) \;=\; \emptyset \end{cases}$$

$$\mathsf{F} \xmapsto{\mathsf{B}} \mathsf{G} \qquad \text{iff} \qquad \begin{cases} \mathcal{R}eq_{\mathsf{B}}(\mathsf{F}) \;=\; \mathcal{O}bs(\mathsf{G}) \cup \mathcal{R}eq_{\mathsf{B}}(\mathsf{G}) \\ \mathcal{R}eq_{\bar{\mathsf{B}}}(\mathsf{G}) \;=\; \mathcal{O}bs(\mathsf{F}) \cup \mathcal{R}eq_{\bar{\mathsf{B}}}(\mathsf{F}). \end{cases}$$

Note that $\mathsf{F} \xmapsto{\mathsf{A}} \mathsf{G}$ (resp., $\mathsf{F} \xmapsto{\mathsf{B}} \mathsf{G}$) implies $\mathsf{F} \xrightarrow{\mathsf{A}} \mathsf{G}$ (resp., $\mathsf{F} \xrightarrow{\mathsf{B}} \mathsf{G}$), but the converse implications are not true in general. Moreover, it is easy to see that any consistent and fulfilling finite compass structure $\mathcal{G} = (\mathbb{I}_N, \mathcal{L})$, with $N \in \mathbb{N}$, satisfies the following properties, and, conversely, any finite structure $\mathcal{G} = (\mathbb{I}_N, \mathcal{L})$, with $N \in \mathbb{N}$, that satisfies the following properties is a consistent and fulfilling compass structure:

- i) for every pair of points $p = (x, y)$ and $q = (y, y+1)$ in $\mathcal{G}$, we have $\mathcal{L}(p) \xmapsto{\mathsf{A}} \mathcal{L}(q)$,
- ii) for every pair of points $p = (x, y)$ and $q = (x, y+1)$ in $\mathcal{G}$, we have $\mathcal{L}(q) \xmapsto{\mathsf{B}} \mathcal{L}(p)$,
- iii) for the lower-left point $p = (0, 1)$ in $\mathcal{G}$, we have $\mathcal{R}eq_{\mathsf{B}}\big(\mathcal{L}(p)\big) = \emptyset$,
- iv) for every upper point $p = (x, N)$ in $\mathcal{G}$, we have $\mathcal{R}eq_{\bar{\mathsf{B}}}\big(\mathcal{L}(p)\big) = \emptyset$ and $\mathcal{R}eq_{\mathsf{A}}\big(\mathcal{L}(p)\big) = \emptyset$.

Now, we can prove Lemma 4.2.

Proof. We first consider the (easier) case of satisfiability with interpretation over finite interval structures; then, we shall deal with the more general case of satisfiability with interpretation over infinite interval structures.

FINITE CASE. In Figure 6, we describe an EXPSPACE non-deterministic procedure that decides whether a given $\mathsf{AB\bar{B}}$ formula is satisfiable over finite labeled interval structures. Below, we prove that such a procedure is sound and complete.

(SOUNDNESS) As for the soundness, we consider a successful computation of the procedure and we show that there is a finite compass structure $\mathcal{G} = (\mathbb{P}_N, \mathcal{L})$ that features φ, where

let φ **be an input formula**

procedure CheckConsistency$(\mathsf{S}, \mathsf{f}, \vec{\mathsf{G}})$
$$\begin{cases} \textbf{for each } \varphi\textbf{-atom } \mathsf{F} \in \mathsf{S} \\ \quad \textbf{do} \quad \begin{cases} \textbf{if } \mathsf{F} \overset{A}{\not\mapsto} \vec{\mathsf{G}} \textbf{ or } \mathsf{f}(\mathsf{F}) \overset{B}{\not\mapsto} \mathsf{F} \\ \quad \textbf{then return false} \end{cases} \\ \textbf{return true} \end{cases}$$

procedure CheckFulfillment(S)
$$\begin{cases} \textbf{for each } \varphi\textbf{-atom } \mathsf{F} \in \mathsf{S} \\ \quad \textbf{do} \quad \begin{cases} \textbf{if } \mathcal{R}eq_{\mathsf{A}}(\mathsf{F}) \neq \emptyset \textbf{ or } \mathcal{R}eq_{\bar{\mathsf{B}}}(\mathsf{F}) \neq \emptyset \\ \quad \textbf{then return false} \end{cases} \\ \textbf{return true} \end{cases}$$

main
$$\begin{cases} \mathsf{N} \leftarrow \textbf{ any value in } \{1, ..., 2^{2^{7|\varphi|}}\} \\ \mathsf{F} \leftarrow \textbf{ any } \varphi\textbf{-atom such that } \mathcal{R}eq_{\mathsf{B}}(\mathsf{F}) = \emptyset \textbf{ and } \varphi \in \mathcal{O}bs(\mathsf{F}) \cup \mathcal{R}eq_{\bar{\mathsf{B}}}(\mathsf{F}) \\ \mathsf{S} \leftarrow \{\mathsf{F}\} \\ \textbf{for } y \leftarrow 1 \textbf{ to } \mathsf{N} \\ \quad \textbf{do} \quad \begin{cases} \mathsf{f} \leftarrow \textbf{ any mapping from } \mathsf{S} \textbf{ to the set of all } \varphi\textbf{-atoms} \\ \vec{\mathsf{G}} \leftarrow \textbf{ any } \varphi\textbf{-atom} \\ \textbf{if not CheckConsistency}(\mathsf{S}, \mathsf{g}, \vec{\mathsf{G}}) \\ \quad \textbf{then return false} \\ \mathsf{S} \leftarrow \{\mathsf{f}(\mathsf{F}) : \mathsf{F} \in \mathsf{S}\} \cup \{\vec{\mathsf{G}}\} \end{cases} \\ \textbf{return CheckFulfillment}(\mathsf{S}) \end{cases}$$

Figure 6: Algorithm for the satisfiability problem over finite structures.

$\mathsf{N} \in \mathbb{N}$ is exactly the value that was guesses at the beginning of the computation. We build such a structure $\mathcal{G}$ inductively on the value of the variable $y \in \{1, ..., \mathsf{N}\}$ as follows.

- If $y = 1$, then we let $\mathcal{G}_1 = (\mathbb{I}_1, \mathcal{L}_1)$, where $\mathcal{L}_1$ maps the unique point of $\mathbb{I}_1$ to the atom F that was guessed at the beginning of the computation. Note that $\mathcal{G}_1$ satisfies the consistency condition of Definition 2.1, but it may not satisfy the fulfillment condition for the relations A and $\bar{B}$.
- If $y > 1$, then assuming that $\mathcal{G}_{y-1} = (\mathbb{I}_{y-1}, \mathcal{L}_{y-1})$ is the consistent (possibly non-fulfilling) compass structure obtained during the $y-1$-th iteration, we define $\mathcal{G}_y = (\mathbb{I}_y, \mathcal{L}_y)$, where:
  - i) $\mathcal{L}_y(p) = \mathcal{L}_{y-1}(p)$ for every point $p = (x', y')$ that belongs to $\mathbb{I}_{y-1}$, namely, such that $0 \leqslant x' < y' < y$;
  - ii) $\mathcal{L}_y(p) = \mathsf{f}\big(\mathcal{L}_{y-1}(q)\big)$ for every pair of points of the form $p = (x, y)$ and $q = (x, y-1)$, with $0 \leqslant x < y - 1$, where f is the function guessed during the y-th iteration;
  - iii) $\mathcal{L}_y(\vec{p}) = \vec{\mathsf{G}}$, where $\vec{p} = (y-1, y)$ and $\vec{\mathsf{G}}$ is the atom guessed during the y-th iteration.

We then define $\mathcal{G}$ to be the structure $\mathcal{G}_{\mathsf{N}}$. Now, knowing that every call to the function **CheckConsistency** was successful, we can conclude that the structure $\mathcal{G}$ satisfies the following two properties:

i) for every pair of points $\mathsf{p} = (\mathsf{x}, \mathsf{y})$ and $\mathsf{q} = (\mathsf{y}, \mathsf{y}+1)$ in $\mathcal{G}$, we have $\mathcal{L}(\mathsf{p}) \xmapsto{\mathsf{A}} \mathcal{L}(\mathsf{q})$,

ii) for every pair of points $\mathsf{p} = (\mathsf{x}, \mathsf{y})$ and $\mathsf{q} = (\mathsf{x}, \mathsf{y}+1)$ in $\mathcal{G}$, we have $\mathcal{L}(\mathsf{q}) \xmapsto{\mathsf{B}} \mathcal{L}(\mathsf{p})$.

Moreover, since the first guessed atom F was such that $\mathcal{R}eq_{\mathsf{B}}(\mathsf{F}) = \emptyset$ and since the call to the function **CheckFulfillment** at the end of the computation was successful, we know that $\mathcal{G}$ satisfies also the following two properties:

iii) for the lower-left point $\mathsf{p} = (0, 1)$, we have $\mathcal{R}eq_{\mathsf{B}}\big(\mathcal{L}(\mathsf{p})\big) = \emptyset$,

iv) for every upper point $\mathsf{p} = (\mathsf{x}, \mathsf{N})$, we have $\mathcal{R}eq_{\bar{\mathsf{B}}}\big(\mathcal{L}(\mathsf{p})\big) = \emptyset$ and $\mathcal{R}eq_{\mathsf{A}}\big(\mathcal{L}(\mathsf{p})\big) = \emptyset$.

By previous arguments, this shows that $\mathcal{G}$ is a consistent and fulfilling compass structure. Finally, since the first guessed atom F was such that $\varphi \in \mathcal{O}bs(\mathsf{F}) \cup \mathcal{R}eq_{\bar{\mathsf{B}}}(\mathsf{F})$, we have that $\mathcal{G}$ features the input formula φ. Proposition 2.2 finally implies that there is a labeled finite interval structure that satisfies φ.

(Completeness) As for completeness, we consider a finite labeled interval structure $\mathcal{S} = (\mathbb{I}_{\mathsf{N}'}, \mathsf{A}, \mathsf{B}, \bar{\mathsf{B}}, \sigma)$ that satisfies φ. By Theorem 3.3, we know that there is a (consistent and fulfilling) compass structure $\mathcal{G} = (\mathbb{I}_{\mathsf{N}}, \mathcal{L})$ of length $\mathsf{N} \leqslant 2^{2^{7|\varphi|}}$ that features φ. We exploit such a structure $\mathcal{G}$ to show that there is a successful computation of the algorithm of Figure 6. To do that, it is sufficient to describe, at each step of the computation where the value of a variable needs to be guessed, which is the right choice for that value. Clearly, at the beginning of the computation, the variable N will take as value exactly the length of the compass structure $\mathcal{G}$. Similarly, the initial value for the variable F is chosen to be the atom $\big\{\mathcal{L}(\mathsf{p})\big\}$ associated with the lower-left point $\mathsf{p} = (0, 1)$. Then, at each iteration of the main loop, we choose the values for f and for $\vec{\mathsf{G}}$ as follows. We assume that, at the y-th iteration, S is exactly the shading associated with the row y in $\mathcal{G}$ (it can be easily proved that this is an invariant of the computation) and, for every atom F in S, we denote by $\mathsf{p}_{\mathsf{F}} = (\mathsf{x}_{\mathsf{F}}, \mathsf{y})$ a generic point along the row y such that $\mathcal{L}(\mathsf{p}_{\mathsf{F}}) = \mathsf{F}$ (such a point exists by assumption). We then choose f to be the function that maps every atom $\mathsf{F} \in \mathsf{S}$ to the atom $\mathsf{f}(\mathsf{F}) = \mathcal{L}(\mathsf{x}_{\mathsf{F}}, \mathsf{y}+1)$. It is routine to prove that the computation that results from the above-defined sequence of guesses is successful.

Infinite case. Figure 7 reports an EXPSPACE non-deterministic procedure that decides whether a given $\mathsf{AB\bar{B}}$ formula is satisfiable over infinite labeled interval structures.

(Soundness) In order to prove that the described procedure is sound, we consider a successful computation of the procedure and we show that there is an infinite periodic compass structure $\mathcal{G} = (\mathbb{P}_{\omega}, \mathcal{L})$ that features φ. The threshold $\widetilde{\mathsf{y}}_0$ and the period $\widetilde{\mathsf{y}}$ of $\mathcal{G}$ are defined to be the values of the corresponding variables that were guessed at the beginning of the computation. As for the binding function $\widetilde{\mathsf{g}}$, we choose any arbitrary mapping $\widetilde{\mathsf{g}}$ from S to $\widetilde{\mathsf{S}}$ such that $\widetilde{\mathsf{g}} \circ \widetilde{\mathsf{f}}$ is the identity on S, where S, $\widetilde{\mathsf{S}}$, and $\widetilde{\mathsf{f}}$ are the values of the corresponding variables at the end of the computation. It now remains to describe the labeling of the finite portion $\mathbb{P}_{\widetilde{\mathsf{y}}_0+\widetilde{\mathsf{y}}-1}$ of $\mathcal{G}$ (note that this labeling uniquely determines the infinite periodic compass structure $\mathcal{G}$). This can be done by following the same construction given in the finite case. Similarly, the fact that $\mathcal{G}$ satisfies the consitency conditions of Definition 2.1 can be proved by exploiting arguments analogous to the finite case. The proof that $\mathcal{G}$ satisfies also the fulfillment condition requires more details. In particular, one can prove, again by exploiting induction on y, that for every row y, with $\widetilde{\mathsf{y}}_0 \leqslant \mathsf{y} < \widetilde{\mathsf{y}}_0 + \widetilde{\mathsf{y}}$, every point $p = (x, \widetilde{y}_0)$, every relation $R \in \{A, \bar{B}\}$, and every R-request $\alpha \in \mathcal{R}eq_R\big(\mathcal{L}(p)\big)$, if $\mathcal{L}(p) = F$ $(\in \widetilde{S})$ and *fulfilled*$[F, R, \alpha]$ is true during the y-th iteration of the main loop, then there exists a point $q = (x', y)$ such that $p\ R\ q$ and $\alpha \in \mathcal{O}bs\big(\mathcal{L}(q)\big)$. Thus, at the end of the computation, since all entries of the variable *fulfilled* are set to **true**, we know that all A-requests and all $\bar{B}$-requests of atoms associated with row $\widetilde{y}_0$ are fulfilled below row $\widetilde{y}_0 + \widetilde{y}$. This shows that $\mathcal{G}$ is a consistent and fulfilling compass structure. As before, one can conclude that $\mathcal{G}$ features the input formula φ and hence there exists an infinite labeled interval structure that satisfies φ.

(Completeness) As for completeness, we consider an infinite labeled interval structure $\mathcal{S} = (\mathbb{I}_\omega, A, B, \bar{B}, \sigma)$ that satisfies φ. By Theorem 3.5, we know that there is a periodic (consistent and fulfilling) compass structure $\mathcal{G} = (\mathbb{I}_\omega, \mathcal{L})$, with threshold $\widetilde{y}_0 < 2^{2^{7|\varphi|}}$, period $\widetilde{y} < 2|\varphi| \cdot 2^{2^{7|\varphi|}} \cdot 2^{2^{7|\varphi|}}$, and binding $\widetilde{g} : \{0, ..., \widetilde{y}_0 + \widetilde{y} - 1\} \to \{0, ..., \widetilde{y}_0 - 1\}$. We exploit such a periodic structure $\mathcal{G}$ to show that there is a successful computation of the algorithm of Figure 7. In particular, at each step of the computation where the value of a variable needs to be guessed, we describe which is the right choice for that value. Clearly, at the beginning of the computation, the variables $\widetilde{y}_0$ and $\widetilde{y}$ will take as values exactly the threshold and the period of the compass structure $\mathcal{G}$. Similarly, the initial value for the variable F is chosen to be the atom $\big\{\mathcal{L}(p)\big\}$ associated with the lower-left point $p = (0, 1)$. Then, at each iteration of one of the two main loops, we choose the values for f and for $\overrightarrow{G}$ as follows. We assume that, at each iteration of one of the two loops, S is the shading associated with the row y in $\mathcal{G}$, where y is the value of the corresponding variable (it can be easily proved that this is an invariant of the computation) and, for every atom F in S, we denote by $p_F = (x_F, y)$ a generic point along the row y such that $\mathcal{L}(p_F) = F$ (such a point exists by assumption). We then choose f to be the function that maps every atom $F \in S$ to the atom $f(F) = \mathcal{L}(x_F, y+1)$. It is routine to prove that the computation that results from the above-defined sequence of guesses is successful. □

let φ **be an input formula**

procedure CheckConsistency$(\mathsf{S}, \mathsf{f}, \overrightarrow{\mathsf{G}})$

(as before)

procedure UpdateFulfillment$(\mathit{fulfilled}, \widetilde{\mathsf{S}}, \widetilde{\mathsf{f}}, \mathsf{S}, \overrightarrow{\mathsf{G}})$

- **for each** φ**-atom** $\mathsf{F} \in \widetilde{\mathsf{S}}$ **and** A**-request** $\alpha \in \mathcal{R}eq_{\mathsf{A}}(\mathsf{F})$
  - **do**
    - **if** $\alpha \in \mathcal{O}bs(\overrightarrow{\mathsf{G}})$
    - **then** $\mathit{fulfilled}[\mathsf{F}, \mathsf{A}, \alpha] \leftarrow$ **true**
- **for each** φ**-atom** $\mathsf{F} \in \widetilde{\mathsf{S}}$ **and** $\bar{\mathsf{B}}$**-request** $\alpha \in \mathcal{R}eq_{\bar{\mathsf{B}}}(\mathsf{F})$
  - **do**
    - **if** $\alpha \in \mathcal{O}bs(\widetilde{\mathsf{f}}(\mathsf{F}))$
    - **then** $\mathit{fulfilled}[\mathsf{F}, \bar{\mathsf{B}}, \alpha] \leftarrow$ **true**

procedure CheckFulfillment$(\mathit{fulfilled}, \widetilde{\mathsf{S}}, \widetilde{\mathsf{f}}, \mathsf{S})$

- **if** $\mathsf{S} \neq \widetilde{\mathsf{S}}$
  - **then return false**
- **for each** φ**-atom** $\mathsf{F} \in \widetilde{\mathsf{S}}$, **relation** $\mathsf{R} \in \{\mathsf{A}, \bar{\mathsf{B}}\}$, **and** R**-request** $\alpha \in \mathcal{R}eq_{\mathsf{R}}(\mathsf{F})$
  - **do**
    - **if not** $\mathit{fulfilled}[\mathsf{F}, \mathsf{R}, \alpha]$
    - **then return false**
- **return true**

main

- $\widetilde{y}_0 \leftarrow$ **any value in** $\left\{1, ..., 2^{2^{7|\varphi|}-1}\right\}$
- $\widetilde{y} \leftarrow$ **any value in** $\left\{1, ..., 2|\varphi| \cdot 2^{2^{7|\varphi|}} \cdot 2^{2^{7|\varphi|}} - 1\right\}$
- $\mathsf{F} \leftarrow$ **any** φ**-atom such that** $\mathcal{R}eq_{\mathsf{B}}(\mathsf{F}) = \emptyset$ **and** $\varphi \in \mathcal{O}bs(\mathsf{F}) \cup \mathcal{R}eq_{\bar{\mathsf{B}}}(\mathsf{F})$
- $\mathsf{S} \leftarrow \{\mathsf{F}\}$
- **for** $y \leftarrow 1$ **to** $\widetilde{y}_0$
  - **do**
    - $\mathsf{f} \leftarrow$ **any mapping from** S **to the set of all** φ**-atoms**
    - $\overrightarrow{\mathsf{G}} \leftarrow$ **any** φ**-atom**
    - **if not CheckConsistency**$(\mathsf{S}, \mathsf{g}, \overrightarrow{\mathsf{G}})$
      - **then return false**
    - $\mathsf{S} \leftarrow \left\{\mathsf{f}(\mathsf{F}) : \mathsf{F} \in \mathsf{S}\right\} \cup \left\{\overrightarrow{\mathsf{G}}\right\}$
- $\widetilde{\mathsf{S}} \leftarrow \mathsf{S}$
- $\widetilde{\mathsf{f}} \leftarrow$ **the identity function on** $\widetilde{\mathsf{S}}$
- **for each** φ**-atom** $\mathsf{F} \in \widetilde{\mathsf{S}}$, **relation** $\mathsf{R} \in \{\mathsf{A}, \bar{\mathsf{B}}\}$, **and** R**-request** $\alpha \in \mathcal{R}eq_{\mathsf{R}}(\mathsf{F})$
  - **do** $\mathit{fulfilled}[\mathsf{F}, \mathsf{R}, \alpha] \leftarrow$ **false**
- **for** $y \leftarrow \widetilde{y}_0 + 1$ **to** $\widetilde{y}_0 + \widetilde{y}$
  - **do**
    - $\mathsf{f} \leftarrow$ **any mapping from** S **to the set of all** φ**-atoms**
    - $\overrightarrow{\mathsf{G}} \leftarrow$ **any** φ**-atom**
    - **if not CheckConsistency**$(\mathsf{S}, \mathsf{g}, \overrightarrow{\mathsf{G}})$
      - **then return false**
    - $\widetilde{\mathsf{f}} \leftarrow \mathsf{f} \circ \widetilde{\mathsf{f}}$
    - $\mathsf{S} \leftarrow \left\{\mathsf{f}(\mathsf{F}) : \mathsf{F} \in \mathsf{S}\right\} \cup \left\{\overrightarrow{\mathsf{G}}\right\}$
    - **UpdateFulfillment**$(\mathit{fulfilled}, \widetilde{\mathsf{S}}, \widetilde{\mathsf{f}}, \mathsf{S}, \overrightarrow{\mathsf{G}})$
- **return CheckFulfillment**$(\mathit{fulfilled}, \widetilde{\mathsf{S}}, \widetilde{\mathsf{f}}, \mathsf{S})$

Figure 7: Algorithm for the satisfiability problem over infinite structures.